Coherent states and rational surfaces

2010 J. Phys. A: Math. Theor. 43 255205

(http://iopscience.iop.org/1751-8121/43/25/255205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 255205 (14pp) doi:10.1088/1751-8113/43/25/255205

Coherent states and rational surfaces

Dorje C Brody[1] **and Eva-Maria Graefe**[1,2]

[1] Department of Mathematics, Imperial College London, London SW7 2AZ, UK
[2] School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

E-mail: d.brody@imperial.ac.uk

Received 19 January 2010, in final form 26 April 2010
Published 27 May 2010
Online at stacks.iop.org/JPhysA/43/255205

Abstract
The state spaces of generalized coherent states associated with special unitary groups are shown to form rational curves and surfaces in the space of pure states. These curves and surfaces are generated by the various Veronese embeddings of the underlying state space into higher dimensional state spaces. This construction is applied to the parameterization of generalized coherent states, which is useful for practical calculations, and provides an elementary combinatorial approach to the geometry of the coherent state space. The results are extended to Hilbert spaces with indefinite inner products, leading to the introduction of a new kind of generalized coherent states.

PACS numbers: 03.65.Ca, 02.40.Tt, 02.40.Hw, 03.65.Sq

1. Introduction

In the present paper we show that the space of generalized coherent states associated with the group $SU(k+1)$ for any $k = 1, 2, \ldots$ can be precisely characterized through the algebraic–geometric concept of a Veronese variety, which concerns certain embeddings of projective spaces into those of higher dimension. This formulation elucidates the geometry of $SU(k+1)$ coherent states in an elementary and visual manner. This previously unobserved link between the geometry of rational curves and surfaces and the theory of generalized coherent states constitutes a striking example of the intrinsic geometrical aspects of quantum theories and may furthermore be useful for practical calculations.

The paper begins with an introduction to M-mode Glauber coherent states and generalized $SU(M)$ coherent states. A rearrangement of terms demonstrates how these two concepts are related. This is followed by a brief introduction to the geometry of complex projective spaces and the associated Fubini–Study metrics. We then discuss the Veronese embeddings of a complex projective line into higher dimensional complex projective spaces, and analyse the geometry of algebraic curves obtained from these embeddings. Further properties of Glauber coherent states are then presented in the style of Geroch (1971), as also used in Field and Hughston (1999) and Brody and Hughston (2000), whereby the connection with the theory of rational curves and Veronese embeddings becomes apparent. Glauber coherent states are then compared with $SU(2)$ coherent states. We show, in particular, how $SU(2)$ coherent states form rational curves in complex projective spaces. This is followed by an analysis of $SU(k+1)$ coherent states for $k = 1, 2, \ldots$, which correspond to algebraic varieties associated with generalized Veronese embeddings. This method permits the explicit parameterization of arbitrary $SU(k+1)$ coherent states in arbitrary dimensions as well as the geometrical description of coherent state spaces in a straightforward manner. We then consider the $SU(1, 1)$ coherent states of Solomon (1971) and Perelomov (1972), and derive the hyperbolic metric induced on the coherent state manifold by the ambient Fubini–Study geometry. Finite-dimensional analogues of $SU(1, k)$ coherent states are then introduced. These coherent states appear naturally in the context of Hilbert spaces endowed with indefinite inner products. Our discussion of coherent states will be primarily focussed upon their geometrical aspects; for a general exposition of coherent states, see, e.g., Klauder and Sudarshan (1968); Klauder and Skagerstam (1985); Perelomov (1986); Zhang *et al* (1990); Berman *et al* (1994) and Vourdas (2006).

2. Quantum coherent states

In quantum mechanics the state of a single-particle system is characterized by a vector in Hilbert space $\mathcal{H}$ equipped with a Hermitian inner product, while observables are represented by self-adjoint operators acting on $\mathcal{H}$. The single-particle Hilbert space may be finite or infinite dimensional. The dynamics of the quantum system described by a Hamiltonian operator $\hat{H}$ is governed by the Schrödinger equation, whose solution is given by the action of the time evolution operator $\hat{U}(t) = \exp(-\mathrm{i}\hat{H}t/\hbar)$ on the initial state, where $\hbar$ denotes the Planck constant.

In many applications of quantum mechanics, subspaces of quantum states possessing certain physical properties are of particular interest. An important example is that of coherent states, which satisfy minimal uncertainty conditions and are such that the 'classical' dynamics of the system is determined by the leading-order dynamics of these states in an expansion in powers of $\hbar$. The coherent state concept in its modern form was introduced by Glauber (1963) in the context of quantum optics for multi-boson systems, which we shall discuss briefly below.

Given a single-particle Hilbert space $\mathcal{H}$, the state space of a general multi-particle system can be constructed as follows. The Hilbert space for a combined system of two particles is the tensor product $\mathcal{H} \otimes \mathcal{H}$, symmetrized for bosons and antisymmetrized for fermions, and similarly for three and more particles. The direct sum of these multi-boson state spaces forms a Fock space

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_s \mathcal{H}) \oplus (\mathcal{H} \otimes_s \mathcal{H} \otimes_s \mathcal{H}) \oplus \cdots. \tag{1}$$

Here $\otimes_s$ denotes the symmetrized tensor product. A convenient basis for the Fock space arising from an M-dimensional single-particle Hilbert space $\mathcal{H}$ is the so-called Fock basis $|n_1 \ldots n_M\rangle$, where n_j denotes the number of particles in the jth basis state of $\mathcal{H}$. A generic state vector $|\xi\rangle$ in the Fock space can then be expressed in the form

$$|\xi\rangle = \sum_{n_1 \cdots n_M} \xi_{n_1 \cdots n_M} |n_1 \cdots n_M\rangle. \tag{2}$$

We define M pairs of ladder operators $\{\hat{A}_j\}$ and $\{\hat{A}_j^\dagger\}$, annihilating or creating a particle in the jth basis state:

$$\begin{cases} \hat{A}_j^\dagger |n_1 \cdots n_M\rangle = \sqrt{n_j + 1}\, |n_1 \cdots n_j{+}1 \cdots n_M\rangle, \\ \hat{A}_j |n_1 \cdots n_M\rangle = \sqrt{n_j}\, |n_1 \cdots n_j{-}1 \cdots n_M\rangle. \end{cases} \tag{3}$$

These operators satisfy the canonical commutation relations $\big[\hat{A}_j, \hat{A}_k^\dagger\big] = \delta_{jk}$ and $[\hat{A}_j, \hat{A}_k] = \big[\hat{A}_j^\dagger, \hat{A}_k^\dagger\big] = 0$, and form the Lie algebra of the Heisenberg–Weyl group. (Note that these creation and annihilation operators should not be confused with the multi-particle creation and annihilation operators used, e.g., in Katriel *et al* 1987.)

The *M*-mode Glauber coherent states $|a\rangle$ can be defined via the action of a displacement operator $\hat{D}(a) = \prod_j \mathrm{e}^{a_j \hat{A}_j^\dagger - a_j^* \hat{A}_j}$ of the Heisenberg–Weyl group on the vacuum state $|0\rangle = |0\cdots 0\rangle$, in which none of the single-particle states is populated:

$$|a\rangle = \hat{D}(a)|0\rangle = \prod_j \mathrm{e}^{a_j \hat{A}_j^\dagger - a_j^* \hat{A}_j}|0\rangle$$
$$= \mathrm{e}^{\frac{1}{2}\sum_j |a_j|^2} \sum_{n_1,\ldots,n_M=0}^{\infty} \left(\prod_{j=1}^{M} \frac{a_j^{n_j}}{\sqrt{n_j!}} \right) |n_1 \cdots n_M\rangle. \tag{4}$$

Such states possess a number of special features and therefore play important roles in various fields of quantum physics. First, they are eigenstates of the annihilation operators, i.e. $\hat{A}_j|a\rangle = a_j|a\rangle$. Second, they form an 'over-complete' set of basis vectors for the symmetric Fock space and constitute a resolution of the identity. (The latter property is related to the fact that projective varieties associated with coherent state manifolds are 'balanced' in the ambient state space manifold; see Donaldson (2001).) Third, they saturate the lower bound of the position–momentum uncertainty relation (where the position and momentum operators are the real and imaginary parts of the annihilation operator, respectively), and thus represent, in some respects, states that are closest to being classical. Finally, under time evolution with a Hamiltonian that is linear in the generators of the Heisenberg–Weyl algebra, an initially coherent state remains coherent, and the expectation values of the position and momentum operators evolve according to the corresponding classical equations of motion. This last property remains valid to leading order in $\hbar$ for an arbitrary Hamiltonian, which constitutes another reason why coherent states are often viewed as representing classical states.

The algebraic characterization of Glauber coherent states presented above has been generalized to systems with arbitrary Lie group structures by Perelomov (1972); see also Radcliffe (1971) and Gilmore (1972). One important example is that of the $SU(2)$ or so-called atomic coherent states. These arise naturally in the context of rotationally invariant systems, where the components $\hat{L}_{x,y,z}$ of the angular momentum operator generate an algebra isomorphic with $\mathfrak{su}(2)$. If the Hamiltonian commutes with the total angular momentum operator $\hat{\mathbf{L}}$, then the Hilbert space of the system is a direct sum of joint eigenspaces of $\hat{H}$ and $\hat{L}^2$, each rotationally irreducible and of dimension $2L + 1$, where $L(L + 1)$ is the appropriate eigenvalue of $\hat{L}^2$. We shall, in what follows, confine our considerations to one such eigenspace. In analogy with the Glauber coherent states (4), the $SU(2)$ coherent states $|\theta, \phi\rangle$ can be defined by the action of the $SU(2)$ displacement operator $\hat{D}(\theta, \phi) = \mathrm{e}^{\mathrm{i}\theta(\hat{L}_x \sin\phi - \hat{L}_y \cos\phi)}$ on the eigenstate $|-L\rangle$ of the angular momentum operator $\hat{L}_z$ corresponding to the lowest eigenvalue:

$$|\theta, \phi\rangle = \hat{D}(\theta, \phi)| - L\rangle = \mathrm{e}^{\mathrm{i}\theta(\hat{L}_x \sin\phi - \hat{L}_y \cos\phi)}| - L\rangle. \tag{5}$$

These atomic coherent states satisfy the minimal uncertainty relations for the angular momentum operators, and their dynamics coincide with those of the corresponding classical states to leading order in $\hbar$.

Interestingly, the Glauber coherent states for an *M*-dimensional single-particle Hilbert space can be constructed as a sum over $SU(M)$ coherent states, as we shall briefly demonstrate below for the two-dimensional case. In accordance with the Schwinger representation of angular momentum (Schwinger 1952), the angular momentum operators may be described in terms of a two-state Heisenberg–Weyl algebra spanned by $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_1^\dagger$, $\hat{A}_2^\dagger$ as

$$L_x = \frac{1}{2}\big(\hat{A}_1^\dagger \hat{A}_2 + \hat{A}_2^\dagger \hat{A}_1\big), \qquad L_y = \frac{1}{2\mathrm{i}}\big(\hat{A}_1^\dagger \hat{A}_2 - \hat{A}_2^\dagger \hat{A}_1\big), \qquad L_z = \frac{1}{2}\big(\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2\big) \tag{6}$$

with the additional restriction that the number of bosons is fixed as $N = 2L$. Thus, we can interpret a $(2L+1)$-dimensional $SU(2)$ system as a two-state system populated with $2L$ bosons, and in the Fock basis the $SU(2)$ coherent states (5) may be expressed in the form

$$\begin{aligned} |\theta,\phi\rangle &= \hat{D}(\theta,\phi)|0,N\rangle = \mathrm{e}^{\mathrm{i}\theta(\hat{L}_x\sin\phi - \hat{L}_y\cos\phi)}|0,N\rangle \\ &= \sum_{j=0}^{N}\sqrt{\binom{N}{j}}\left(\cos\frac{1}{2}\theta\right)^{N-j}\left(\sin\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi}\right)^{j}|j,N-j\rangle. \end{aligned} \tag{7}$$

On the other hand, the Glauber coherent states for two modes can be rewritten as

$$\begin{aligned} |a_1,a_2\rangle &= \mathrm{e}^{\frac{1}{2}(|a_1|^2+|a_2|^2)}\sum_{n_1,n_2=0}^{\infty}\frac{a_1^{n_1}}{\sqrt{n_1!}}\frac{a_2^{n_2}}{\sqrt{n_2!}}|n_1,n_2\rangle \\ &= \mathrm{e}^{\frac{1}{2}(|a_1|^2+|a_2|^2)}\sum_{N=0}^{\infty}\sum_{j=0}^{N}\frac{a_1^{N-j}a_2^{j}}{\sqrt{(N-j)!j!}}|j,N-j\rangle \\ &= \mathrm{e}^{\frac{1}{2}(|a_1|^2+|a_2|^2)}\sum_{N=0}^{\infty}\frac{1}{\sqrt{N!}}\sum_{j=0}^{N}\sqrt{\binom{N}{j}}a_1^{N-j}a_2^{j}|j,N-j\rangle. \end{aligned} \tag{8}$$

Hence, identifying $a_1 = c\cos\frac{1}{2}\theta$ and $a_2 = c\sin\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi}$ with $c \in \mathbb{C}-\{0\}$, we see that the terms in the Glauber coherent states with a constant boson number N are proportional to the $SU(2)$ coherent states. The relation between M-mode Glauber coherent states and $SU(M)$ coherent states for arbitrary M can be established by an analogous construction.

In view of the important role played by coherent states in various physical applications, we shall analyse in further detail the geometry of the subspaces of the quantum state space spanned by the coherent states. Before proceeding, however, we first briefly review the Fubini–Study geometry of the space of pure states, and the concepts of rational curves and Veronese embeddings.

3. Fubini–Study geometry of quantum state space

In quantum mechanics the expectation value of an observable $\hat{H}$ in a state $|z\rangle \in \mathcal{H}$, which represents the outcome of measurements, is given by $\langle z,\hat{H}z\rangle/\langle z,z\rangle$. This is invariant under the overall complex phase shift $|z\rangle \to \lambda|z\rangle$, $\lambda \in \mathbb{C}-\{0\}$. Hence two Hilbert space vectors only differing by a complex scale factor represent the same physical state. We can therefore eliminate the overall physically irrelevant degree of freedom by considering the space of equivalence class modulo the identification $|z\rangle \sim \lambda|z\rangle$. The resulting projective Hilbert space is the space of pure quantum states, which, in the case of a finite, $(n+1)$-dimensional system is the complex projective n-space $\mathbb{CP}^n$. The geometry of the space of pure states, the Fubini–Study geometry, characterizes important aspects of the physical behaviour of a system, and will be reviewed in the following.

We begin by remarking that the complex projective space $\mathbb{CP}^n$ is the quotient space of a real $2n+1$ sphere by the circle group $U(1)$, i.e. $\mathbb{CP}^n = S^{2n+1}/U(1)$. This can be seen as follows. Consider a complex projective line $\mathbb{CP}^1$, i.e. the quotient space of $\mathbb{C}^2$ under the identification $(w_1,w_2) \sim (\lambda w_1,\lambda w_2)$ for all $\lambda \in \mathbb{C}-\{0\}$. In other words, $\mathbb{CP}^1$ is the space of rays through the origin of $\mathbb{C}^2$; two points in $\mathbb{C}^2$ define the same point of $\mathbb{CP}^1$ iff they lie on the same complex line through the origin. Now if we first normalize the overall scale of $\mathbb{C}^2$, we obtain the three-sphere $|w_1|^2 + |w_2|^2 = 1$; if we further identify vectors differing only by a phase factor, then we obtain the Hopf fibration $S^3/U(1) = S^2$. Thus, in real terms the complex projective line can be viewed as a two-sphere S^2. This is also evident from the fact that the intersection of a sphere and a line through the origin of $\mathbb{C}^2$ is a real circle. Analogous constructions clearly exist in higher dimensions. Thus, for $\mathbb{C}^3$ we normalize the scale to obtain a five-sphere $|w_1|^2 + |w_2|^2 + |w_3|^2 = 1$, whence the complex projective plane $\mathbb{CP}^2$, i.e. the space of rays through the origin of $\mathbb{C}^3$, is obtained via the Hopf fibration $S^5/U(1) = \mathbb{CP}^2$. More generally, the space of rays in $\mathbb{C}^{n+1}$ is just the quotient $\mathbb{CP}^n = S^{2n+1}/U(1)$ since every line through the origin of $\mathbb{C}^{n+1}$ intersects the unit sphere in a circle.

The points of the complex projective space $\mathbb{CP}^n$ are often conveniently represented by homogeneous coordinates $(z^0, z^1, z^2, \ldots, z^n)$, with a redundant complex degree of freedom. To specify the geometry of $\mathbb{CP}^n$ induced by the ambient Euclidean geometry of $\mathbb{C}^{n+1}$, consider the inner product of neighbouring points. Writing ds for the line element and $\langle\,,\rangle$ for the inner product in $\mathbb{C}^{n+1}$, we have

$$\frac{\langle \bar{z}, z + \mathrm{d}z\rangle\langle \bar{z} + \mathrm{d}\bar{z}, z\rangle}{\langle \bar{z}, z\rangle\langle \bar{z} + \mathrm{d}\bar{z}, z + \mathrm{d}z\rangle} = \cos^2 \frac{1}{2}\,\mathrm{d}s. \tag{9}$$

Solving this for ds and retaining terms of quadratic order, we obtain the Fubini–Study line element

$$\mathrm{d}s^2 = 4\frac{\langle \bar{z}, z\rangle\langle \mathrm{d}\bar{z}, \mathrm{d}z\rangle - \langle \bar{z}, \mathrm{d}z\rangle\langle z, \mathrm{d}\bar{z}\rangle}{\langle \bar{z}, z\rangle^2}. \tag{10}$$

An alternative derivation of the Fubini–Study metric employs the fact that complex projective spaces possess Kählerian structures. In particular, the Kähler potential K for $\mathbb{CP}^n$, in terms of inhomogeneous coordinates $(\zeta^1, \zeta^2, \ldots, \zeta^n) = (z^1/z^0, z^2/z^0, \ldots, z^n/z^0)$ on an appropriate coordinate patch $z^0 \neq 0$, is given by

$$K = 4\ln(1 + \bar{\zeta}_j\zeta^j). \tag{11}$$

Thus, by use of the standard definition

$$\mathrm{d}s^2 = \frac{\partial^2 K}{\partial\zeta^i\partial\bar{\zeta}_j}\,\mathrm{d}\zeta^i\,\mathrm{d}\bar{\zeta}_j \tag{12}$$

of a Kähler metric, we obtain the familiar expression (Kobayashi and Nomizu 1969) for the Fubini–Study metric:

$$\mathrm{d}s^2 = 4\frac{(1 + \bar{\zeta}_j\zeta^j)(\mathrm{d}\bar{\zeta}_j\,\mathrm{d}\zeta^j) - (\bar{\zeta}_j\,\mathrm{d}\zeta^j)(\zeta^j\,\mathrm{d}\bar{\zeta}_j)}{(1 + \bar{\zeta}_j\zeta^j)^2}. \tag{13}$$

The quantum mechanical transition probabilities are thus measured in terms of the Fubini–Study geodesic distances between the states. The expression for the metric will be used in what follows to determine the induced metrics of the various subspaces of the Fubini–Study manifold. In particular, we now turn to the discussion of the Veronese embedding in the Fubini–Study manifolds and derive the geometry induced by the embedding.

4. Veronese embeddings and rational curves

We have seen that the M-mode Glauber coherent states can be viewed as consisting of a combination of $SU(M)$ coherent states over different particle numbers. As we shall indicate later, the state space of $SU(M)$ coherent states arises from the Veronese embedding, the concept of which we shall briefly introduce here. The Veronese variety is concerned with the embedding of a complex projective space into higher dimensional complex projective spaces; in particular, it is an embedding of $\mathbb{CP}^k$ in $\mathbb{CP}^{k(k+3)/2}$ possessing certain properties (Nomizu 1976). Here we review the properties of this embedding for $k = 1$, that is, $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ and its generalizations $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$, which will be shown to characterize $SU(2)$ coherent states.

Let (s, t) be the homogeneous coordinates of a point in $\mathbb{CP}^1$. The image point in $\mathbb{CP}^2$ under the Veronese embedding then has the homogeneous coordinates $(s^2, \sqrt{2}st, t^2)$. Thus, the image of $\mathbb{CP}^1$ in the complex projective plane $\mathbb{CP}^2$ forms a conic curve $\mathcal{C}$ characterizing the solution to a quadratic equation. Because $\mathbb{CP}^1$ in real terms is a two-sphere, this nonsingular one-to-one correspondence between points on $\mathbb{CP}^1$ and points on $\mathcal{C}$ implies that the conic $\mathcal{C}$ is a topological sphere. The metric on $\mathcal{C}$ induced by the ambient Fubini–Study metric of $\mathbb{CP}^2$ can be calculated as follows. We substitute the homogeneous coordinates $(z^1, z^2, z^3) = (s^2, \sqrt{2}st, t^2)$ of $\mathbb{CP}^2$ into formula (10) for the line element, and perform the same calculation for the metric of $\mathbb{CP}^1$ in terms of the homogeneous coordinates $(z^1, z^2) = (s, t)$. A short calculation then yields

$$\mathrm{d}s^2_{\mathcal{C}} = 2\,\mathrm{d}s^2_{\mathbb{CP}^1}, \tag{14}$$

that is, the metric of the conic $\mathcal{C}$ is just twice the metric of $\mathbb{CP}^1$. If we fix the scale so that the radius of $\mathbb{CP}^1 = S^2$ is 1, as we have done implicitly in (9), then in real terms the conic $\mathcal{C}$ is a two-sphere of radius $\sqrt{2}$.

The Veronese embedding of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ can be generalized in a natural way to an embedding of the form $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ such that if (s, t) denotes the homogeneous coordinates for $\mathbb{CP}^1$, then the correspondence

$$(s, t) \hookrightarrow \left(s^n, \sqrt{\binom{n}{1}}s^{n-1}t, \sqrt{\binom{n}{2}}s^{n-2}t^2, \ldots, \sqrt{\binom{n}{n-1}}st^{n-1}, t^n\right) \tag{15}$$

defines the homogeneous coordinates of the image in $\mathbb{CP}^n$. The image for $n = 3$ is a twisted cubic curve (cf Wood 1913), for $n = 4$, a rational quartic curve (cf Telling 1936), for $n = 5$, a rational quintic curve, for $n = 6$, a rational sextic curve, and so on. Thus, the generalized Veronese embedding of $\mathbb{CP}^1$ in $\mathbb{CP}^n$ defines a rational curve in $\mathbb{CP}^n$ (an algebraic curve in $\mathbb{CP}^n$ has topological dimension 2 and thus represents a surface, and rational curves are characterized by the fact that the corresponding surfaces have zero genus); the significance of these rational curves with respect to the structures of elementary quantum spin systems is discussed in detail in Brody and Hughston (2001).

Using representation (15), it is not difficult to specify the geometry of these rational curves. Since direct computation of the metric is cumbersome, we proceed by first calculating the Kähler potential (cf Stanley 1993). We note that the Kähler potential for $\mathbb{CP}^1$ is $K_{\mathbb{CP}^1} = 4\ln(1 + |t/s|^2)$. This follows from (11) by setting $\zeta = t/s$ for the inhomogeneous coordinate of $\mathbb{CP}^1$. We now use (15) and follow the same procedure to derive the Kähler potential for the rational curve $\mathcal{R}$ in $\mathbb{CP}^n$. A short calculation then yields

$$K_{\mathcal{R}} = 4\ln(1 + |t/s|^2)^n, \tag{16}$$

which shows that the metric for the rational curve $\mathcal{R} \in \mathbb{CP}^n$ is n times the metric of a two-sphere with unit radius. It follows that the rational curve $\mathcal{R} \in \mathbb{CP}^n$ in real terms is a two-sphere with radius $\sqrt{n}$ and constant curvature $2/n$.

5. The geometry of Glauber coherent states

To analyse the geometry of the Glauber coherent states and their relation to $SU(M)$ coherent states, it is convenient to characterize Fock space operations in the style of Geroch (1971).

For this purpose we write $\xi \in \mathbb{C}$, and ξ^a for an element of $\mathcal{H}$ where the abstract index a labels the components of the single-particle state in an arbitrary chosen orthonormal basis set. An element of $\mathcal{H} \otimes_s \mathcal{H}$ can then be written $\xi^{ab} = \xi^{(ab)}$, and so on, where round brackets denote symmetrization over tensor indices. A generic state vector $|\xi\rangle$ in Fock space can be written in the form

$$|\xi\rangle = (\xi, \xi^a, \xi^{ab}, \xi^{abc}, \ldots). \tag{17}$$

The inner product of a pair of states $|\xi\rangle$ and $|\eta\rangle$ in Fock space is then

$$\langle\eta|\xi\rangle = \bar{\eta}\xi + \bar{\eta}_a\xi^a + \bar{\eta}_{ab}\xi^{ab} + \cdots. \tag{18}$$

Hence, the norm of a state is $\|\xi\|^2 = \langle\xi|\xi\rangle$, assumed finite.

A state $|\xi\rangle$ in Fock space can be augmented by a particle in the state $\sigma^a \in \mathcal{H}$ via the action of the creation operator $\hat{A}^\dagger_\sigma$, given by

$$\hat{A}^\dagger_\sigma|\xi\rangle = (0, \xi\sigma^a, \sqrt{2}\xi^{(a}\sigma^{b)}, \sqrt{3}\xi^{(ab}\sigma^{c)}, \ldots). \tag{19}$$

Note that the general creation operator $\hat{A}^\dagger_\sigma$ can be viewed as a linear superposition of the elementary creation operators $\{\hat{A}^\dagger_j\}$ introduced in (3). Similarly, the annihilation operator $\hat{A}_\tau$ acts as follows:

$$\hat{A}_\tau|\xi\rangle = (\bar{\tau}_c\xi^c, \sqrt{2}\bar{\tau}_c\xi^{ac}, \sqrt{3}\bar{\tau}_c\xi^{abc}, \ldots). \tag{20}$$

Formulae (19) and (20) imply the canonical commutation relations $\left[\hat{A}^\dagger_\sigma, \hat{A}^\dagger_{\sigma'}\right] = [\hat{A}_\tau, \hat{A}_{\tau'}] = 0$ and $\left[\hat{A}^\dagger_\sigma, \hat{A}_\tau\right] = (\bar{\tau}_a\sigma^a)\mathbb{1}$.

A general multi-particle quantum state $|\xi\rangle$ in Fock space is fully characterized by an analytic function on the single-particle Hilbert space $\mathcal{H}$ (Bargmann 1961). Specifically, the state $|\xi\rangle$ can be fully recovered from the function

$$\Psi(\eta) = \bar{\xi} + \bar{\xi}_a\eta^a + \tfrac{1}{\sqrt{2!}}\bar{\xi}_{ab}\eta^a\eta^b + \tfrac{1}{\sqrt{3!}}\bar{\xi}_{abc}\eta^a\eta^b\eta^c + \cdots \tag{21}$$

for $\eta^a \in \mathcal{H}$. If function (21) is of the exponential form $\Psi(\eta) = \exp(\bar{\xi}_a\eta^a)$, the corresponding state is a Glauber coherent state (4). In this case all the multi-particle components depend upon just one single-particle state, and in the abstract index notation a coherent state has the form

$$|\xi\rangle = \left(1, \xi^a, \tfrac{1}{\sqrt{2!}}\xi^a\xi^b, \tfrac{1}{\sqrt{3!}}\xi^a\xi^b\xi^c, \cdots\right). \tag{22}$$

Because the norm $\|\xi\|$ of a state $|\xi\rangle$ is not physically observable, we may projectivize the Fock space $\mathcal{F}$. The projectivized coherent states then form a submanifold of the projective Fock space. The geometry of the space of Glauber coherent states has been investigated in the literature, and the results can be summarized as follows.

Theorem (Provost and Vallee 1980, Field and Hughston 1999)**.** *The geometry of the coherent-state submanifold of the projective Fock space induced by the ambient Fubini–Study metric is flat, i.e. complex Euclidean.*

Proof. The Fubini–Study metric on the projective Fock space is

$$\mathrm{d}s^2 = 4\left[\frac{\langle \mathrm{d}\xi|\mathrm{d}\xi\rangle}{\langle\xi|\xi\rangle} - \frac{\langle\xi|\mathrm{d}\xi\rangle\langle\mathrm{d}\xi|\xi\rangle}{\langle\xi|\xi\rangle^2}\right]. \tag{23}$$

For a coherent state (22), a straightforward calculation shows that $\langle\xi|\xi\rangle = \exp(\bar{\xi}_a\xi^a)$, $\langle\xi|\mathrm{d}\xi\rangle = \exp(\bar{\xi}_a\xi^a)\bar{\xi}_a\,\mathrm{d}\xi^a$ and $\langle\mathrm{d}\xi|\mathrm{d}\xi\rangle = \exp(\bar{\xi}_a\xi^a)[\mathrm{d}\bar{\xi}_a\,\mathrm{d}\xi^a + (\bar{\xi}_a\,\mathrm{d}\xi^a)(\xi^a\,\mathrm{d}\bar{\xi}_a)]$. Substituting these expressions into (23), we find that the line element is given by $\mathrm{d}s^2 = 4\,\mathrm{d}\bar{\xi}_a\,\mathrm{d}\xi^a$. □

The above proof is equivalent to the first of the three proofs given in Field and Hughston (1999). We also remark that Provost and Vallee (1980) established this result for the case of one-mode Glauber coherent states.

From the algebraic definition of coherent states, the flatness of the Weyl–Heisenberg group manifold seems natural, but from the geometric point of view, this result is at first surprising, since a linear superposition of a pair of coherent states is incoherent, i.e. the complex projective line passing through a pair of coherent states in the projective Fock space lies outside the coherent-state submanifold (except for the two intersection points). The 'linearity' of the submanifold of Glauber coherent states, i.e. the flatness of the induced metric, must be understood with the caveat that although ξ^a and $\lambda\xi^a$ for $\lambda \in \mathbb{C} - \{0\}$ represent the *same* single-particle state vector, the coherent states arising from ξ^a and $\lambda\xi^a$ represent *different* multi-particle state vectors (the inner product $\langle\xi|\lambda\xi\rangle = \langle\xi|\xi\rangle^{\lambda}$ of the associated coherent states agrees with $\langle\xi|\xi\rangle$ iff $\lambda = 1$). Thus, the Glauber coherent-state submanifold of the projective Fock space is endowed with the Euclidean geometry of the underlying single-particle Hilbert space, not the Fubini–Study geometry of the single-particle state space. However, the situation is somewhat different for the atomic coherent states.

6. Rational curves and atomic coherent states

As discussed above, the $SU(M)$ coherent states can be viewed as the N-particle component of an M mode Glauber coherent state (22), suitably renormalized:

$$|\xi\rangle = \frac{\xi^a\xi^b\cdots\xi^c}{(\bar{\xi}_a\xi^a)^{N/2}}. \tag{24}$$

By virtue of the normalization in (24), $SU(M)$ coherent states, unlike Glauber coherent states, are such that a pair of Hilbert space vectors ξ^a and $\lambda\xi^a$ representing the same state vector also represent the same $SU(M)$ coherent state. Thus, $SU(M)$ coherent states do not inherit the Euclidean geometry of the underlying Hilbert space.

In the case of the *atomic* or $SU(2)$ coherent states, the underlying single-particle Hilbert space is two dimensional. Introducing spherical variables for the homogeneous coordinates, we write $(s,t) = \left(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}\phi}\right)$. The atomic coherent state (24) then lies in the $(N+1)$-dimensional subspace of symmetric state vectors in the 2^N-dimensional Hilbert space. Writing $\{|j, N-j\rangle\}_{j=0,\ldots,N}$ for the basis elements of this Hilbert space, we recover the familiar expression (7) for the $SU(2)$ coherent state. Clearly, the coefficients in the $SU(2)$ coherent state (7) are in bijective correspondence with the components of the rational curve (15) in $\mathbb{CP}^N$. We thus conclude that for each N, the $SU(2)$ coherent states form a rational curve $\mathcal{R}$ in the associated projective state space $\mathbb{CP}^N$.

The geometry of the space of $SU(2)$ coherent states is therefore equivalent to that of the rational curve discussed above. In terms of the spherical parameterization the inhomogeneous coordinate of $\mathbb{CP}^1$ is $\zeta = \tan\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi}$, from which the Fubini–Study metric can be calculated via (13). A short calculation then gives the standard Riemannian metric for the unit sphere: $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. Moreover, from (16), the metric of the $(N+1)$-component $SU(2)$ coherent state space is given by

$$\mathrm{d}s^2_{\mathfrak{su}(2)} = N(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{25}$$

This agrees with the result obtained in Provost and Vallee (1980). The coherent-state manifold may be regarded as a classical phase space, with a constant curvature of $2/N$.

As mentioned above, there exists an elegant geometric characterization of the various spin states in terms of the properties of rational curves and their osculating hyperspaces, the spin states being determined by the various intersection points of these surfaces (Brody and Hughston 2001).

7. Generalized Veronese embeddings as $SU(k+1)$ coherent states

We now consider the extension of the $SU(2)$ atomic coherent states characterized by rational curves to the general $SU(k+1)$ coherent states. Recall that a Veronese variety is defined by an embedding of $\mathbb{CP}^k$ in $\mathbb{CP}^{k(k+3)/2}$. For each k this embedding can be generalized in a natural manner such that various state spaces of $SU(k+1)$ coherent states can be generated systematically. We begin with the analysis of $SU(3)$ coherent states.

For $k=2$ the embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5$ can be constructed as follows. Let (s,t,u) be the homogeneous coordinates of a point in $\mathbb{CP}^2$. Then the homogeneous coordinates of the image point in $\mathbb{CP}^5$ are $(s^2, \sqrt{2}st, t^2, \sqrt{2}tu, u^2, \sqrt{2}us)$. This defines a rational quadratic surface (a real four-dimensional simply connected smooth manifold) in $\mathbb{CP}^5$ with the topology of a $\mathbb{CP}^2$.

As in the case of $\mathbb{CP}^1$, the Veronese embedding of $\mathbb{CP}^2$ can also be generalized to embeddings of the form $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^{N(N+3)/2}$, $N=2,3,\ldots$, such that the homogeneous coordinates of the image of (s,t,u) are defined by the trinomial expansion. For example, when $N=3$, the homogeneous coordinates of the image point in $\mathbb{CP}^9$ are $(s^3, \sqrt{3}s^2t, \sqrt{3}st^2, t^3, \sqrt{3}t^2u, \sqrt{3}tu^2, u^3, \sqrt{3}u^2s, \sqrt{3}us^2, \sqrt{6}stu)$.

We now determine the induced metric on the rational surfaces by calculating the Kähler potential. Writing $(\zeta^1, \zeta^2) = (t/s, u/s)$ for the inhomogeneous coordinates of $\mathbb{CP}^2$, the Kähler potential of the generalized Veronese variety $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^{N(N+3)/2}$ is

$$K = 4\ln(1+\bar{\zeta}_1\zeta^1+\bar{\zeta}_2\zeta^2)^N. \tag{26}$$

Thus, the induced metric on these Veronese varieties is that of a complex projective plane, multiplied by the scale factor N.

The foregoing discussion shows that for each value of N the embedding $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^{N(N+3)/2}$ defines a manifold of $SU(3)$ coherent states. In terms of the usual spherical coordinates, we can write

$$(s,t,u) = \left(\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\varphi, \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\varphi\,\mathrm{e}^{\mathrm{i}\xi}, \cos\tfrac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\eta}\right), \tag{27}$$

and substitution of this expression into the trinomial expansion for each N then yields the parametric representation of the corresponding $\frac{1}{2}(N^2+3N+2)$-level $SU(3)$ coherent state:

$$|\theta,\varphi,\xi,\eta\rangle = \sum_{k=0}^{N}\sum_{l=0}^{k}\sqrt{\frac{N!}{l!(N-k)!}}\left(\sin\frac{1}{2}\theta\cos\frac{1}{2}\varphi\right)^l \times\left(\sin\frac{1}{2}\theta\sin\frac{1}{2}\varphi\,\mathrm{e}^{\mathrm{i}\xi}\right)^{k-l}\left(\cos\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\eta}\right)^{N-k}|k,l\rangle. \tag{28}$$

The state space of $SU(3)$ coherent states may be regarded as a classical phase space, and geometrically this is just a complex projective plane $\mathbb{CP}^2$ with curvature $6/N$ (see also Gnutzmann and Kuś (1998) for a detailed analysis of the $SU(3)$ coherent states).

Similarly, for each k we may consider the Veronese embedding $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^{k(k+3)/2}$ and its generalizations. Then the validity of the following statement should be evident.

Proposition. *The totality of $SU(k+1)$ coherent states is characterized by the family of generalized Veronese varieties*

$$\mathbb{CP}^k \hookrightarrow \mathbb{CP}^{\frac{1}{k!}(N+1)(N+2)\cdots(N+k)-1}, \qquad (N=2,3,\ldots). \tag{29}$$

Further, for each k and N, the induced metric on the SU(k+1) coherent state manifold is that of a $\mathbb{CP}^k$ *scaled by N; hence, the curvature is scaled by* $1/N$.

The standard spherical representation for $\mathbb{CP}^k$ can thus be used to parameterize the coherent-state manifolds, which may be useful in various applications such as passage to the classical limit (cf Gnutzmann and Kuś 1998, Graefe *et al* 2008, Trimborn *et al* 2008, Yaffe 1982) or, conversely, geometric quantization (cf Rawnsley 1977).

8. Coherent states on hyperbolic domains

Counterparts of the $SU(k+1)$ coherent states, associated with the noncompact group $SU(1,k)$, also have various applications in physics. We shall derive the metrics of the relevant state spaces by the methods outlined above. To this end, we consider a Hilbert space equipped with an indefinite inner product $\langle\bar{z},w\rangle_- = -\bar{z}_0 w^0 + \bar{z}_j w^j$ on $\mathbb{C}^{k+1}$, with the isometry group $SU(1,k)$. Restricting consideration to the subspace for which, say, $\langle\bar{z},z\rangle_- < 0$, and forming as before the space of rays through the origin, we obtain the complex hyperbolic space $\mathbb{CH}^k$. Here, the analogue of the Hopf fibration $S^1 \to S^{2k+1} \to \mathbb{CP}^k$ is $S^1 \to Q^{2k+1} \to \mathbb{CH}^k$, where Q^{2k+1} is the pseudo-sphere $\langle\bar{z},z\rangle_- = -1$.

In the indefinite case, we have

$$\frac{\langle\bar{z},z+\mathrm{d}z\rangle_-\langle\bar{z}+\mathrm{d}\bar{z},z\rangle_-}{\langle\bar{z},z\rangle_-\langle\bar{z}+\mathrm{d}\bar{z},z+\mathrm{d}z\rangle_-} = \cosh^2\frac{1}{2}\mathrm{d}s \tag{30}$$

in place of (9), and thus the metric on the complex hyperbolic space $\mathbb{CH}^k$ is

$$\mathrm{d}s^2 = -4\frac{\langle\bar{z},z\rangle_-\langle\mathrm{d}\bar{z},\mathrm{d}z\rangle_- - \langle\bar{z},\mathrm{d}z\rangle_-\langle z,\mathrm{d}\bar{z}\rangle_-}{\langle\bar{z},z\rangle_-^2}. \tag{31}$$

In inhomogeneous coordinates, the hyperbolic metric (31) assumes the more familiar form (Kobayashi and Nomizu 1969)[3]:

$$\mathrm{d}s^2 = 4\frac{(1-\bar{\zeta}_j\zeta^j)(\mathrm{d}\bar{\zeta}_j\,\mathrm{d}\zeta^j) + (\bar{\zeta}_j\,\mathrm{d}\zeta^j)(\zeta^j\,\mathrm{d}\bar{\zeta}_j)}{(1-\bar{\zeta}_j\zeta^j)^2}. \tag{32}$$

That (32) reduces to (31) can be verified by substituting $\mathrm{d}\zeta^j = (z^0\mathrm{d}z^j - z^j\mathrm{d}z^0)/(z^0)^2$ and $\mathrm{d}\bar{\zeta}^j = (\bar{z}_0\,\mathrm{d}\bar{z}_j - \bar{z}_j\,\mathrm{d}\bar{z}_0)/(\bar{z}_0)^2$ into (32) and rearranging terms. Alternatively, (32) may be deduced from the expression $K = 4\ln(1-\bar{\zeta}_j\zeta^j)$ for the Kähler potential. Up to a scale factor, (32) is the unique Riemannian metric on $\mathbb{CH}^k$ such that the action of $U(1,k)$ is isometric.

Let us consider in more detail the geometry of the complex hyperbolic line $\mathbb{CH}^1$. The appropriate metric on this space is sometimes known as the Bergman metric since it can be obtained from of the Bergman kernel (Bergman 1970). We shall briefly explain this idea in view of its relevance to the theory of generalized coherent states. We have restricted the states to the region where $\langle\bar{z},z\rangle_- < 0$, i.e. $-\bar{z}_0 z^0 + \bar{z}_1 z^1 < 0$. Passing to the inhomogeneous coordinate, the inequality $\bar{\zeta}\zeta < 1$ defines the unit (Poincaré) disc $\mathcal{D}$ in the complex plane. Consider the Hilbert space $\mathcal{L}^2(\mathcal{D})$ of complex-valued functions square-integrable with respect to the Lebesgue measure $\mathrm{d}x\mathrm{d}y$ on $\mathcal{D}$, where $\zeta = x+\mathrm{i}y$, and the complex orthonormal functions on $\mathcal{D}$ defined by

$$\phi_n(\zeta) = \left(\frac{n}{\pi}\right)^{\frac{1}{2}}\zeta^{n-1}. \tag{33}$$

[3] Note the error in the sign of the second term in the expression for the hyperbolic metric in Kobayashi and Nomizu (1969).

A direct calculation shows that $\{\phi_n\}$ forms an orthonormal basis for $\mathcal{L}^2(\mathcal{D})$:

$$\iint_{\mathcal{D}} \mathrm{d}x\,\mathrm{d}y\,\overline{\phi_n(\zeta)}\phi_m(\zeta) = \frac{\sqrt{nm}}{\pi}\int_0^1 r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\, r^{n+m-2}\,\mathrm{e}^{-\mathrm{i}\theta(n-m)} = \frac{2\sqrt{nm}}{n+m}\,\delta_{nm} = \delta_{nm}. \quad (34)$$

The kernel K_B corresponding to this orthonormal set is defined by

$$K_B(\zeta,\bar\chi) = \sum_{n=1}^{\infty}\phi_n(\zeta)\overline{\phi_n(\chi)}. \quad (35)$$

The function $K_B(\zeta,\bar\chi)$, known as the Bergman kernel, satisfies the identity

$$g(\zeta) = \iint_{\mathcal{D}} \mathrm{d}x\,\mathrm{d}y\,K_B(\zeta,\bar\chi)g(\chi), \quad (36)$$

(where $\chi = x + \mathrm{i}y$) for any smooth function $g \in \mathcal{L}^2(\mathcal{D})$. This kernel function generally diverges for a real orthonormal basis, but always converges uniformly in the complex case (Helgason 1978). In the present case, substituting (33) into (35), we obtain

$$K_B(\zeta,\bar\chi) = \frac{1}{\pi(1-\zeta\bar\chi)^2}. \quad (37)$$

The kernel function K_B associated with a domain $\mathcal{D}$ in the complex plane naturally determines a Riemannian metric on $\mathcal{D}$, called the Bergman metric, via the prescription

$$\mathrm{d}s^2 = \frac{\partial^2}{\partial\bar\zeta\,\partial\zeta}\ln K_B(\zeta,\bar\zeta)\,\mathrm{d}\bar\zeta\,\mathrm{d}\zeta. \quad (38)$$

Two salient properties of this metric are invariance under conformal transformations and monotonicity in the sense that if $\mathcal{D}' \subset \mathcal{D}$, then the associated line elements satisfy $\mathrm{d}s' > \mathrm{d}s$. In the present example this line element, up to a scale factor, is given by

$$\mathrm{d}s^2 = 4\,\frac{\mathrm{d}\bar\zeta\,\mathrm{d}\zeta}{(1-\bar\zeta\zeta)^2}, \quad (39)$$

which agrees with (32) for $k = 1$.

The standard formulation of $SU(1,1)$ coherent states, due to Solomon (1971) and Perelomov (1972), is closely related to the geometric quantization (cf Odzijewicz 1992) of the Poincaré disc $\mathcal{D}$ defined by $\bar\zeta\zeta < 1$, and is based upon the infinite-dimensional Hilbert space $\mathcal{L}^2(\mathcal{D})$ spanned by the orthonormal functions (33). Specifically, Perelomov (1972) defines, for any $|\xi| < 1$, the generic coherent state as the Hilbert space vector

$$|\xi\rangle = \sum_{n=1}^{\infty}\sqrt{n}\,\xi^{n-1}\phi_n(\zeta) \quad (40)$$

(here we consider the lowest order state in the representation). In this case, the Hilbert space $\mathcal{L}^2(\mathcal{D})$ is equipped with a positive-definite inner product, which projectively defines the Fubini–Study metric. Moreover,

Theorem. *The metric of the standard $SU(1,1)$ coherent state submanifold of the projective Fock space induced by the ambient Fubini–Study metric is hyperbolic.*

Proof. The Fubini–Study metric on the projective Hilbert space assumes the form

$$\mathrm{d}s^2 = 4\left[\frac{\langle \mathrm{d}\xi|\mathrm{d}\xi\rangle}{\langle\xi|\xi\rangle} - \frac{\langle\xi|\mathrm{d}\xi\rangle\langle\mathrm{d}\xi|\xi\rangle}{\langle\xi|\xi\rangle^2}\right]. \quad (41)$$

For the coherent state (40), a simple calculation shows that $\langle\xi|\xi\rangle = (1-\bar{\xi}\xi)^{-2}$, $\langle\xi|\,\mathrm{d}\xi\rangle = 2\bar{\xi}\,\mathrm{d}\xi(1-\bar{\xi}\xi)^{-3}$ and $\langle\mathrm{d}\xi|\,\mathrm{d}\xi\rangle = 2(1+2\bar{\xi}\xi)(1-\bar{\xi}\xi)^{-4}\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi$. Substituting these into (41), we find that the line element is $\mathrm{d}s^2 = 8(1-\bar{\xi}\xi)^{-2}\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi$. □

This hyperbolic geometry of the $SU(1,1)$ coherent state space is well known (cf Perelomov 1986), but to our knowledge its explicit derivation from the ambient Fubini–Study geometry, as illustrated here, has not previously appeared in the relevant literature. More generally, for $\bar{\xi}_j\xi^j < 1$ we define the $SU(1,k)$-analogue of the $SU(1,1)$ coherent state (40) by

$$|\xi\rangle = (1, \sqrt{2}\xi^i, \sqrt{3}\xi^i\xi^j, \sqrt{4}\xi^i\xi^j\xi^l, \ldots). \tag{42}$$

Then, by a simple calculation, $\langle\xi|\xi\rangle = (1-\bar{\xi}_j\xi^j)^{-2}$, $\langle\xi|\,\mathrm{d}\xi\rangle = 2(1-\bar{\xi}_j\xi^j)^{-3}\bar{\xi}_j\,\mathrm{d}\xi^j$ and $\langle\mathrm{d}\xi|\,\mathrm{d}\xi\rangle = 2(1-\bar{\xi}_j\xi^j)^{-3}\,\mathrm{d}\bar{\xi}_j\,\mathrm{d}\xi^j + 6(1-\bar{\xi}_j\xi^j)^{-4}(\bar{\xi}_j\,\mathrm{d}\xi^j)(\xi^j\,\mathrm{d}\bar{\xi}_j)$. Substituting these into (41), we obtain the line element

$$\mathrm{d}s^2 = 8\,\frac{(1-\bar{\xi}_j\xi^j)(\mathrm{d}\bar{\xi}_j\,\mathrm{d}\xi^j) + (\bar{\xi}_j\,\mathrm{d}\xi^j)(\xi^j\,\mathrm{d}\bar{\xi}_j)}{(1-\bar{\xi}_j\xi^j)^2}, \tag{43}$$

which is just twice the metric of the original $\mathbb{CH}^k$.

9. Atomic coherent states for indefinite Hilbert spaces

We shall now construct generalized coherent states associated with the group $SU(1,k)$ by defining Veronese-type maps for indefinite Hilbert spaces. This result provides finite-dimensional $SU(1,k)$-analogues of the $SU(k+1)$ coherent states defined on a $(k+1)$-dimensional Hilbert space with an indefinite Hermitian inner product of the Pontryagin type (Pontryagin 1944). The foregoing discussion of the Veronese embedding might convey the preliminary impression that the distinction between the state spaces $\mathbb{CP}^k$ and $\mathbb{CH}^k$ is merely formal and insignificant. However, there are essential differences between these two cases. In the present context, for instance, a Veronese construction for an embedding of the form, say, $\mathbb{CH}^1 \hookrightarrow \mathbb{CH}^2$ does not exist. To see this, let (s,t) denote the homogeneous coordinates of $\mathbb{CH}^1$. Then $-\bar{s}s + \bar{t}t = -1$; squaring this, we obtain $(\bar{s}s)^2 - 2\bar{s}s\bar{t}t + (\bar{t}t)^2 = +1$. On the other hand, if $(s^2, \sqrt{2}st, t^2)$ were the homogeneous coordinates of a point in $\mathbb{CH}^2$, then we would have $-(\bar{s}s)^2 + 2\bar{s}s\bar{t}t + (\bar{t}t)^2 = -1$, contradicting the previous equation. Alternatively, note that a point on $\mathbb{CH}^2$ can be parameterized in the form

$$(z^0, z^1, z^2) = \left(\cosh\tfrac{1}{2}\tau, \sinh\tfrac{1}{2}\tau\cos\theta\,\mathrm{e}^{\mathrm{i}\alpha}, \sinh\tfrac{1}{2}\tau\sin\theta\,\mathrm{e}^{\mathrm{i}\beta}\right), \tag{44}$$

whereas a point on $\mathbb{CH}^1$ can be expressed in the form $(z^0, z^1) = \left(\cosh\tfrac{1}{2}\tau, \sinh\tfrac{1}{2}\tau\,\mathrm{e}^{\mathrm{i}\phi}\right)$. Hence, $\mathbb{CH}^1$ cannot be embedded into $\mathbb{CH}^2$ via a Veronese-type construction.

Nevertheless, one can define a Veronese-type map $\mathbb{CH}^k \hookrightarrow \mathfrak{M}$ for a certain hyperbolic Kähler manifold $\mathfrak{M}$ having a signature structure distinct from that of any $\mathbb{CH}^k$. For example, the embedding $(s,t) \hookrightarrow (s^3, \sqrt{3}s^2t, \sqrt{3}st^2, t^3)$ of $\mathbb{CH}^1$ defines an $SU(1,1)$ 'coherent' state within the state space of an $SU(2,2)$ system, rather than an $SU(1,3)$ system. Since the state space of the $SU(2,2)$ system is not a $\mathbb{CH}^3$, the interpretation of this embedding is somewhat different from that of the previously considered map relating to $SU(2)$ coherent states. Nevertheless, it is of interest, at least from a mathematical viewpoint, to formulate a concept of $SU(1,k)$ coherent states applicable to Hilbert spaces with indefinite inner products.

We consider here only the case $k = 1$. Using the standard parameterization $(s,t) = \left(\cosh\tfrac{1}{2}\tau, \sinh\tfrac{1}{2}\tau\,\mathrm{e}^{\mathrm{i}\phi}\right)$ for the homogeneous coordinates on $\mathbb{CH}^1$, we obtain

$$|\tau,\phi\rangle = \sum_{j=0}^{N}\sqrt{\binom{N}{j}}\left(\cosh\frac{1}{2}\tau\right)^j\left(\sinh\frac{1}{2}\tau\,\mathrm{e}^{\mathrm{i}\phi}\right)^{N-j}|j\rangle, \tag{45}$$

which can be regarded as an embedding of $SU(1,1)$ 'coherent' states within the state space of an $SU((N+1)/2,(N+1)/2)$ system if N is odd, and an $SU(1+N/2,N/2)$ system if N is even. For any N, the metric of this submanifold can easily be calculated from the Bergman kernel, and the result is a hyperbolic metric of the form (39), scaled by the factor N.

We find therefore that the construction of finite-dimensional $SU(1,1)$ coherent states is entirely feasible even though the standard algebraic definition precludes such an object because $SU(1,1)$ has no finite-dimensional unitary representations. This demonstrates the flexibility in our geometric construction of coherent state spaces. The coherent state (45) and its generalizations may prove useful in the various applications of the Pontryagin–Kreĭn spaces.

10. Discussion

Our consideration has been focussed upon the metric properties of the various coherent state spaces; the algebraic geometry of these spaces is rather intricate and will be discussed elsewhere. Here, we merely mention that the $SU(k+1)$ coherent states for $k=1,2,\ldots$ possess a natural 'hierarchical' structure arising from the Veronese subvarieties associated with a series of embeddings of the form

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5 \hookrightarrow \mathbb{CP}^{20} \hookrightarrow \mathbb{CP}^{230} \hookrightarrow \cdots \tag{46}$$

for $SU(2)$ and $SU(3)$, and its generalizations (e.g. $\mathbb{CP}^3 \hookrightarrow \mathbb{CP}^9 \hookrightarrow \mathbb{CP}^{54} \hookrightarrow \mathbb{CP}^{1539} \hookrightarrow \cdots$ for $SU(4)$; $\mathbb{CP}^4 \hookrightarrow \mathbb{CP}^{14} \hookrightarrow \mathbb{CP}^{119} \hookrightarrow \mathbb{CP}^{7497} \hookrightarrow \cdots$ for $SU(5)$, and so on). Thus, for example, within the space of $SU(3)$ coherent states corresponding to each value of N there is a subanifold of $SU(2)$ coherent states, and so on. By means of generalized Veronese embeddings, the nesting of these $SU(k+1)$ coherent states can be succinctly described by elementary combinatorics. In particular, the natural metric structures of all these subspaces are of the Fubini–Study type.

Acknowledgment

The authors thank D Blasius, E J Brody and L P Hughston for useful comments and stimulating discussions.

References

Bargmann V 1961 On a Hilbert space of analytic functions and an associated integral transform *Commun. Pure. Appl. Math.* **14** 187–214

Bergman S 1970 *The Kernel Function and Conformal Mapping* (Providence, RI: American Mathematical Society)

Berman G P, Bulgakov E N and Holm D D 1994 *Crossover-time in Quantum Boson and Spin Systems* (Berlin: Springer)

Brody D C and Hughston L P 2000 Classical fields as statistical states *Twistor Newsl.* **45** 40–3

Brody D C and Hughston L P 2001 Geometric quantum mechanics *J. Geom. Phys.* **38** 19–53

Donaldson S K 2001 Scalar curvature and projective embeddings I *J. Diff. Geom.* **59** 479–522

Field T R and Hughston L P 1999 The geometry of coherent states *J. Math. Phys.* **40** 2568–83

Geroch R 1971 An approach to quantization of general relativity *Ann. Phys.* **62** 582–9

Gilmore R 1972 Geometry of symmetrised states *Ann. Phys.* **74** 391–463

Glauber R J 1963 Coherent and incoherent states of radiation field *Phys. Rev.* **131** 2767–88

Gnutzmann S and Kuś M 1998 Coherent states and the classical limit on irreducible SU_3 representations *J. Phys. A: Math. Gen.* **31** 9871–96

Graefe E M, Korsch H J and Niederle A E 2008 Mean-field dynamics of a non-Hermitian Bose–Hubberd dimer *Phys. Rev. Lett.* **101** 150408

Helgason S 1978 *Differential Geometry, Lie Groups, and Symmetric Spaces* (San Diego: Academic)

Katriel J, Solomon A I, D'Ariano G and Rasetti M 1987 Multiphoton squeezed states *J. Opt. Soc. Am.* **4** 1728–36

Klauder J R and Skagerstam B S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)

Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)

Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry* vol 2 (New York: Wiley)

Nomizu K 1976 A characterisation of the Veronese varieties *Nagoya Math. J.* **60** 181–8

Odzijewicz A 1992 Coherent states and geometric quantisation *Commun. Math. Phys.* **150** 385–413

Perelomov A M 1972 Coherent states for arbitrary Lie group *Commun. Math. Phys.* **26** 222–36

Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)

Pontryagin L S 1944 Hermitian operators in spaces with indefinite metrics *Bull. Acad. Sci. URSS Ser. Math.* **8** 243–80

Provost J P and Vallee G 1980 Riemannian structure on manifolds of quantum states *Commun. Math. Phys.* **76** 289–301

Radcliffe J M 1971 Some properties of coherent spin states *J. Phys. A: Gen. Phys.* **4** 313–23

Rawnsley J H 1977 Coherent states and Kähler manifolds *Quart. J. Math. Oxford* **28** 403–15

Schwinger J 1952 On angular momentum *USAEC Report NYO-3071*; Reprinted in: Biedenharn L C van Dam H (eds) 1965 Quantum Theory of Angular Momentum (New York: Academic)

Solomon A I 1971 Group theory of superfluidity *J. Math. Phys.* **12** 390–4

Stanley A 1993 The geometric phase on Kähler manifolds *J. Phys. A: Math. Gen.* **26** L1223–8

Telling H G 1936 *The Rational Quartic Curve in Space of Three and Four Dimensions* (Cambridge: Cambridge University Press)

Trimborn F, Witthaut D and Korsch H J 2008 Exact number conserving phase-space dynamics of the M-site Bose–Hubbard model *Phys. Rev.* A **77** 043631

Vourdas A 2006 Analytic representations in quantum mechanics *J. Phys. A: Math. Gen.* **39** R65–141

Wood P W 1913 *The Twisted Cubic with Some Account of the Metrical Properties of the Cubical Hyperbola* (Cambridge: Cambridge University Press)

Yaffe L G 1982 Large *N* limits as classical mechanics *Rev. Mod. Phys.* **54** 407–35

Zhang W M, Feng D H and Gilmore R 1990 Coherent states: theory and some applications *Rev. Mod. Phys.* **62** 867–927